# Math 255A Lecture 20 Notes

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# 1 Analytic Fredholm Theory

## 1.1 Analytic Fredholm theory

**Theorem 1.1** (analytic Fredholm theory)**.** *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $T(z) \in \mathcal{L}(B_1, B_2)$ for $z \in \Omega$ be a family for Fredholm operators depending holomorphically on $z$; that is $T : z \mapsto T(z)$ is holomorphic with respect to the operator norm on $L(B_1, B_2)$. Assume that there exists $z_0 \in \Omega$ such that $T(z_0) : B_1 \to B_2$ is invertible. Then there exists a set $\Sigma \subseteq \Omega$ having no limit point in $\Omega$ such that for all $z \in \Omega \setminus \Sigma$, the operator $T(z) : B_1 \to B_2$ is is bijective.*

*Proof.* Notice that $z \mapsto \operatorname{ind}(T(z))$ is constant, so $\operatorname{ind}(T(z)) = \operatorname{ind}(T(z_0)) = 0$ for all $z \in \Omega$. Let $z_1 \in \Omega$, and write $n_0(z_1) = \dim(\ker(T(z_1))) = \dim(\operatorname{coker}(T(z_1)))$. Consider the Grushin operator for $T(z_1)$:

$$\mathcal{P}^{z_1} = \begin{bmatrix} T(z_1) & R_-(z_1) \\ R_+(z_1) & 0 \end{bmatrix} : B_1 \oplus \mathbb{C}^{n_0(z_1)} \to B_2 \oplus \mathbb{C}^{n_0(z_1)},$$

which is invertible. There exists a connected open neighborhood $N(z_1) \subseteq \Omega$ of $z_1$ such that for $z \in N(z_1)$, the operator

$$\mathcal{P}^{z_1}(z) = \begin{bmatrix} T(z) & R_-(z_1) \\ R_+(z_1) & 0 \end{bmatrix}$$

is bijective and depends holomorphically on $z \in N(z_1)$.

Let

$$\mathcal{E}^{z_1}(z) = (\mathcal{P}^{z_1}(z))^{-1} = \begin{bmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{bmatrix}$$

be the inverse of $\mathcal{P}^{z_1}(z)$, depending holomorphically on $z \in N(z_1)$. We claim that for $z \in N(z_1$, we have $T(z) : B_1 \to B_2$ is bijective if and only if $E_{-+}(z) : \mathbb{C}^{n_0(z_1)} \to \mathbb{C}^{n_0(z_1)}$ is bijective.

$$\begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

so we get $TE + R_-E_- = I$ and $TE_+ + E_-E_{-+}$. If $E_{-+}^{-1}$ exists, then $TE_+E_{-+}^{-1} = R_-$, so $T(E - E_+E_{-+}^{-1}E_-) = I$. Thus, $T$ is surjective, so because $T$ is Fredholm of index 0, $T$ is bijective, and $T^{-1} = E - E_+E_{-+}^{-1}E_-$. The converse is checked similarly.

$E_{-+}$ is a holomorphic function with values in $n_0(z_1) \times n_0(z_1)$ matrices. So it is bijective iff $\det(E_{-+}) \neq 0$. We have that either $\det(E_{-+}(z)) = 0$ on $N(z_1)$ or $\det(E_{-+}) \neq 0$ in a deleted neighborhood of $z_1$. Let $\Omega_1 = \{z \in \Omega : T(z') \text{ is invertible } \forall z' \neq z \text{ near } z\}$, and let $\Omega_2 = \{z \in \Omega : T(z') \text{ is not invertible } \forall z' \neq z \text{ near } z\}$. Then $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1, \Omega_2$ are open. $\Omega_1 \neq \varnothing$, so $\Omega_2 = \varnothing$, and thus the set $\Sigma = \{z \in \Omega : T(z) \text{ is not invertible}\}$ is a closed set with only isolated points. $\square$

## 1.2 Behavior of inverses near singularities

**Remark 1.1.** We have $z \mapsto T(z)^{-1}$ is holomorphic on $\Omega \setminus \Sigma$. Consider the behavior of $T(z)^{-1}$ near $w \in \Sigma$. Write $T(z)^{-1} = E(z) - E_+(z)E_{-+}^{-1}(z)E_-(z)$. Then $E_{-+}(z)^{-1}$ has a pole at $z = w$ (because we are dividing by the determinant, which may has zeros of at most finite multiplicity), so

$$E_{-+}(z)^{-1} = \frac{R_{N_0}}{(z-w)^{N_0}} + \cdots + \frac{R_{-1}}{z-w} + \mathrm{Hol}(z).$$

Here, $\mathrm{rank}(R_j) \leq n_0$. It follows that $z \mapsto T(z)^{-1}$ has a poleof order $N_0$ at $z = w$:

$$T(z)^{-1}(z) = \frac{A_{-N_0}}{(z-w)^{N_0}} + \cdots + \frac{A_{-1}}{(z-w)} + Q(z),$$

where $Q(z)$ is holomorphic in a neighborhood of $w$ and takes values in $\mathcal{L}(B_2, B_1)$. The operators $A_{-N_0}, \dots, A_{-1} \in \mathcal{L}(B_2, B_1)$ can be expressed in terms of $R_{-N_0}, \dots, R_{-1}$ and $E_\pm^{(j)}(w)$ and are of finite rank.

**Definition 1.1.** The **spectrum** of $T : B_1 \to B_2$ is

$$\mathrm{Spec}(T) = \{z \in \mathbb{C} : T - zI \text{ is not invertible}\}.$$

Analytic Fredholm theory shows that if $T$ is Fredholm, then $\mathrm{Spec}(T)$ consists of isolated points.